On Prioritized 2-tuple Ordered Weighted Averaging Operators

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Abstract. This paper deals with linguistic aggregation problems where there exists a prioritization relationship over attributes. We propose a prioritized 2-tuple ordered weighted averaging (PTOWA) operator and study its properties. We then use this operator and a TOWA operator to aggregate satisfactions of attributes for alternatives.
Keywords : Multi-attribute decision making, linguistic terms, PTOWA operator.

1 Introduction

In multi-attribute decision making (MADM), due to the complexity and uncertainty of the objective things, as well as the fuzziness of the human mind, some attributes are suitable to be evaluated in the form of language[1]-[7]. For example, when evaluating the comprehensive qualities of the students or the performance of cars, the decision makers prefer to use 'excellent', 'good' and 'poor' to give an evaluation. For linguistic information aggregation, various linguistic aggregation operators have been proposed, including linguistic OWA operator [1], induced-linguistic OWA operator [2], linguistic WOWA operator [3], etc. In the aggregation process of these operators, the results do not exactly match any of the initial linguistic terms. Therefore, an approximation process has to be developed to express the result in the initial expression domain, but leads to the loss of information and lack of precision. Herrera and Martínez [4] presented an analysis method based upon 2-tuple for linguistic aggregation. Then they proposed 2-tuple weighted average (TWA) operator and 2-tuple ordered weighted averaging (TOWA) operator [4], and successfully applied the TOWA operator to multigranular hierarchical linguistic contexts in a multi-expert decision making problem [5]. Many achievements have been taken in MADM by using these linguistic aggregation operators.

It is important to see that the above linguistic aggregation operators have the ability to trade off between attributes. While in some situations where there exists a prioritization relationship over the attributes, we do not want to allow this kind of compensation. Yager studied this kind of problem where decision information is described by real numbers. He pointed out that the importance

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weights of lower priority attributes were based on the satisfaction of alternative to the higher priority attributes [8]. Based on this idea, Yager proposed the prioritized average (PA) operator [9] and the prioritized ordered weighted averaging (POWA) operator [10]. Wei and Tang [12] introduced two averaging operators, a generalized PA operator and a generalized POWA operator. In the case with one attribute in each priority category, the two operators reduce to the PA operator and the POWA operator proposed by Yager.

Motivated by the above-mentioned studies, we consider linguistic aggregation problems where there exists a prioritization relationship over the attributes. This paper is structured as follows. In section 2, we shall make a brief review of 2-tuple and its related operators. In section 3, we propose a prioritized 2-tuple ordered weighted averaging (PTOWA) operator and discuss its properties. We then use this operator and a TOWA operator to aggregate satisfactions of attributes by alternatives. The paper is concluded in section 4.

2 2-tuple Linguistic Representation Model and Related Operators

For MADM problems with some qualitative attributes, we need to use a linguistic term set to describe the decision information. Herrera and Martínez [4] introduced a finite and totally ordered discrete linguistic term set: $S = \{s_\alpha | \alpha = 0, 1, \cdots, \tau\}$, whose cardinality value is odd. For example, a set of seven linguistic terms S could be

$$S = \{s_0 = \text{extremely poor},\ s_1 = \text{very poor},\ s_2 = \text{poor},\ s_3 = \text{fair},\\ s_4 = \text{good},\ s_5 = \text{very good},\ s_6 = \text{extremely good}\}.$$

Furthermore, Herrera defined 2-tuple to aggregate linguistic information.

Definition 1. [4] Let $S = \{s_0, s_1, \cdots, s_\tau\}$ be a linguistic term set , then the 2-tuple can be obtained by the translation function θ:

$$\theta :\ S \to S \times [-0.5, 0.5),\ \theta(s_i) = (s_i, 0),\ \text{for any } s_i \in S. \tag{1}$$

Definition 2. [4] Let $S = \{s_0, s_1, \cdots, s_\tau\}$ be a linguistic term set, $s_i \in S$ and $\beta \in [0, \tau]$, a value representing the result of a symbolic aggregation operation, then the 2-tuple can be obtained with the following function:

$$\Delta :\ [0, \tau] \to S \times [-0.5, 0.5),\ \Delta(\beta) = (s_i, \alpha) = \begin{cases} s_i, & i = \text{round}(\beta) \\ \alpha = \beta - i, & i \in [-0.5, 0.5), \end{cases} \tag{2}$$

where round $(\cdot)$ is the usual round operation.

Definition 3. [4] Let $S = \{s_0, s_1, \cdots, s_\tau\}$ be a linguistic term set, $s_i \in S$ and (s_i, α) be a 2-tuple. There is always a Δ^{-1} function such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, \tau]$:

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, \tau], \Delta^{-1}(s_i, \alpha) = i + \alpha = \beta. \tag{3}$$

Let (s_i, α_1) and (s_j, α_2) be two 2-tuples. Then they should have the properties as follows:

(1) There exists an order: if $i > j$ then (s_i, α_1) is bigger than (s_j, α_2); if $i = j$ then

a) if $\alpha_1 = \alpha_2$, then (s_i, α_1) and (s_j, α_2) represent the same information;

b) if $\alpha_1 > \alpha_2$, then (s_i, α_1) is bigger than (s_j, α_2);

c) if $\alpha_1 < \alpha_2$, then (s_i, α_1) is smaller than (s_j, α_2).

(2) There exists a negative operator: $\mathrm{Neg}(s_i, \alpha) = \Delta(\tau - (\Delta^{-1}(s_i, \alpha)))$, where (s_i, α) is an arbitrary 2-tuple, $\tau + 1$ is the cardinality of S, $S = \{s_0, s_1, \cdots, s_\tau\}$.

(3) There exists a minimization and a maximization operator:
$\max\{(s_i, \alpha_1), (s_j, \alpha_2)\} = (s_i, \alpha_1)$, $\min\{(s_i, \alpha_1), (s_j, \alpha_2)\} = (s_j, \alpha_2)$, if $(s_i, \alpha_1) \geq (s_j, \alpha_2)$.

Definition 4. [4] Let $\{(b_1, \alpha_1), (b_2, \alpha_2) \cdots, (b_n, \alpha_n)\}$ be a set of 2-tuples, the 2-tuple ordered weighted averaging (TOWA) operator is defined as

$$\mathrm{TOWA}\{(b_1, \alpha_1), (b_2, \alpha_2), \cdots, (b_n, \alpha_n)\} = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^*\right), \tag{4}$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is the related weighting vector of TOWA operator, such that $w_j \geq 0$, $\sum\limits_{j=1}^{n} w_j = 1$. β_j^* is the jth largest of the values β_i and $\beta_i = \Delta^{-1}(b_i, \alpha_i)$, $i = 1, 2, \cdots n$.

3 PTOWA Operator

For a linguistic MADM problem, we assume that we have a collection of attributes $C = \{C_1, C_2, \cdots, C_n\}$ and there is a prioritization between the attributes expressed by the linear ordering $C_1 > C_2 > \cdots > C_n$. For any alternative x and attribute C_j, we assume that $C_j(x) \in S(x \in X)$ indicates the satisfaction of attribute C_j by alternative x, where $S = \{s_0, s_1, \cdots, s_\tau\}$ is a linguistic term set and τ is an even.

For each attribute, we transform $C_j(x)$ into a 2-tuple, denoted by a_j. According to the prioritization relationship between attributes and the satisfaction a_j, we first obtain the importance weighting vector $u = (u_1, u_2, \cdots, u_n)^T$ of the attributes. For each attribute we assume T_j is its 2-tuple weight. T_j is defined as

$$\text{(i) } T_1 = (s_\tau, 0); \quad \text{(ii) } T_j = \min\{T_{j-1}, a_{j-1}\}, \quad j = 2, 3, \cdots, n. \tag{5}$$

Transform T_j into its equivalent value, and then we get the normalized importance weights

$$u_j = \frac{\Delta^{-1}(T_j)}{\sum\limits_{j=1}^{n} \Delta^{-1}(T_j)}, \quad j = 1, 2 \cdots, n. \tag{6}$$

Now, we obtain the importance weighting vector $u = (u_1, u_2, \cdots, u_n)^T$ of the attributes which reflects the prioritization relationship. For a given alternative x, when using TOWA operator to aggregation its satisfaction to each attribute we must be able to consider the importance weight associated with each attribute. Yager [11] suggested an approach to performing this type of aggregation by using OWA operator. We now used this approach for the case using TOWA operator.

We consider the situation when we start with a weighting vector $w = (w_1, w_2, \cdots, w_n)^T$, such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, of the TOWA operator. We modify these weights $w_j, j = 1, 2 \cdots, n$, to include the weighting vector $u = (u_1, u_2, \cdots, u_n)^T$ of the prioritized attributes. Yager [11] and Torra [3] suggested an approach to obtain these modified weights. They suggested modeling a BUM function, a mapping $f : [0, 1] \to [0, 1]$ satisfying $f(0) = 0, f(1) = 1$, and $f(x) \geq f(y)$ if $x > y$, as a piecewise linear function. It is suggested that the function f interpolates the points $(\frac{i}{n}, \sum_{j<i} w_j)$. With this, we can obtain

$$f(x) = \sum_{k=1}^{j-1} w_k + w_j(nx - (j-1)), \quad \frac{j-1}{n} \leq x \leq \frac{j}{n}. \tag{7}$$

Using this function we can obtain the modified weights $v_j (j = 1, 2, \cdots n)$. We assume $ind(j)$ is the index of the jth largest of a_j. Thus $a_{ind(j)}$ is the jth largest of a_j and $u_{ind(j)}$ is its associated importance weight. Let $R_0 = 0$, $R_j = \sum_{k=1}^{j} u_{ind(k)}$. Then we can calculate the modified weights v_j by

$$v_j = f(R_j) - f(R_{j-1}), \quad j = 1, 2 \cdots, n. \tag{8}$$

The modified weights $v_j (j = 1, 2, \cdots n)$ take into account both the w_j and individual importance weights u_j of the attributes. We now use the modified weights to aggregate the satisfactions of attributes by an alternative. We define a function as follows.

Definition 5. Let $a_j = (b_j, \alpha_j)$ $(j = 1, 2 \cdots, n)$ be the satisfactions of attributes C_j by an alternative, and there is a prioritization between the attributes expressed by the ordering $C_1 > C_2 > \cdots > C_n$. The prioritized 2-tuple ordered weighted averaging (PTOWA) operator is defined as

$$\mathrm{PTOWA}\{(b_1, \alpha_1), (b_2, \alpha_2), \cdots, (b_n, \alpha_n)\} = \Delta\left(\sum_{j=1}^{n} v_j \Delta^{-1}\left(a_{ind(j)}\right)\right), \tag{9}$$

where $a_{ind(j)}$ represents the jth largest of a_j , $v = (v_1, v_2, \cdots, v_n)^T$ is the related weighting vector of PTOWA operator satisfying $v_j \geq 0 (j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} v_j = 1$, and v can be obtained by Eq. (8).

For convenience of notation, we denote PTOWA$\{(b_1,\alpha_1),(b_2,\alpha_2),\cdots,(b_n,\alpha_n)\}$ $= (\tilde{b},\tilde{\alpha})$. We can easily prove that the PTOWA operator satisfies the following properties.

Proposition 1. (Boundedness) Let $\{(b_1,\alpha_1),(b_2,\alpha_2),\cdots,(b_n,\alpha_n)\}$ be a set of 2-tuples, then we have $\min\limits_j \{(b_j,\alpha_j)\} \le (\tilde{b},\tilde{\alpha}) \le \max\limits_j \{(b_j,\alpha_j)\}$.

Proposition 2. (Idempotency) Let $\{(b_1,\alpha_1),(b_2,\alpha_2),\cdots,(b_n,\alpha_n)\}$ be a set of 2-tuples, if $(b_j,\alpha_j)=(b,\alpha)$, $j=1,2,\cdots,n$. Then we obtain $(\tilde{b},\tilde{\alpha})=(b,\alpha)$.

In the preceding we consider the situation that there is a prioritization between the attributes expressed by the linear ordering $C_1 > C_2 > \cdots > C_n$. Here we assume that the collection $C=\{C_1,C_2,\cdots,C_n\}$ of attributes is partitioned into q distinct categories, $H_1,H_2,\cdots,H_q$ such that $H_i=\{C_{i1},C_{i2},\cdots,C_{in_i}\}$. Here C_{ij} are the attributes in category H_i, $C=\bigcup\limits_{i=1}^{q} H_i$ and $n=\sum\limits_{i=1}^{q} n_i$. We assume a prioritization between these categories $H_1 > H_2 > \cdots > H_n$. The attributes in the class H_i have a higher priority than those in H_k if $i<k$. We assume that for any alternative $x\in X$, we have for each attribute C_{ij} a linguistic term $C_{ij}(x)\in S$ indicating its satisfaction to attribute C_{ij}.

We now give a method to aggregate the satisfactions of attributes by alternative x based on the PTOWA operator and the TOWA operator:

1) Aggregate the satisfactions of each category H_i based on the TOWA operator. For each attribute, we transform $C_{ij}(x)$ into a 2-tuple, denoted by a_{ij}. We associate each priority class H_i a TOWA weighting vector $W_i=(w_{i1},w_{i2},\cdots,w_{in_i})^T$, such that $w_{ij}\ge 0$ and $\sum\limits_{j=1}^{n_i} w_{ij}=1$. Using this we calculate the aggregation value a_i of each category H_i: $a_i=\text{TOWA}(a_{i1},a_{i2},\cdots,a_{in_i})=\Delta\left(\sum\limits_{j=1}^{n_i} w_{ij}\beta_{ij}^*\right)$,

where β_{ij}^* is the jth largest of the values β_{ik} and $\beta_{ik}=\Delta^{-1}(a_{ik})$, $k=1,2\cdots,n_i$.

2) Calculate the importance weight of each category H_i. We assume T_j is its 2-tuple weight. T_j is defined as $T_1=(s_\tau,0)$; $T_j=\min\{T_{j-1},a_{j-1}\}$, $j=2,3\cdots q$.

Transform T_j into its equivalent value, and then we get the normalized importance weights $u_j=\frac{\Delta^{-1}(T_j)}{\sum_{j=1}^{q}\Delta^{-1}(T_j)}$, $j=1,2\cdots,q$.

3) Calculate the PTOWA aggregation value for alternative x:

$$\text{PTOWA}(a_1,a_2,\cdots,a_q)=\Delta\left(\sum_{j=1}^{q} v_j\Delta^{-1}\left(a_{ind(j)}\right)\right),$$

where $v=(v_1,v_2,\cdots,v_q)^T$ is the related weighting vector of PTOWA operator. Then we can use the PTOWA aggregation value to rank the alternatives.

Example 1. Consider the following prioritized collection of attributes : $H_1=\{C_{11},C_{12}\}$, $H_2=\{C_{21}\}$, $H_3=\{C_{31},C_{32},C_{33}\}$. We assume there is a prioritization ordering $H_1 > H_2 > H_3$ between these categories and the linguistic term set S is defined as

$$S = \{s_0 = \text{extremely poor},\ s_1 = \text{very poor},\ s_2 = \text{poor},\ s_3 = \text{fair},\\ s_4 = \text{good},\ s_5 = \text{very good},\ s_6 = \text{extremely good}\}.$$

Assume for alternative x we have
$C_{11}(x) = s_3$, $C_{21}(x) = s_4$, $C_{22}(x) = s_6$, $C_{31}(x) = s_3$, $C_{32}(x) = s_4$, $C_{33}(x) = s_1$.

We now using the above method to aggregate the satisfactions of attributes for alternative x. We associate with each priority class H_i an OWA weighting vector W_i as follow: $W_1 = (1)$, $W_2 = (0.5, 0.5)$, $W_3 = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$. For priority class H_i, by sept 1), we can get the TOWA aggregation values $a_1 = (s_3, 0)$, $a_2 = hboxTOWA\{(s_4, 0), (s_6, 0)\} = (s_5, 0)$ and $a_3 = \text{TOWA}\{(s_3, 0), (s_4, 0), (s_1, 0)\} = (s_2, 0.83)$.

By step 2), we get $T_1 = (s_6, 0)$, $T_2 = \min\{T_1, a_1\} = (s_3, 0)$, $T_3 = \min\{T_2, a_2\} = (s_3, 0)$. With this we have $u_1 = 0.5$, $u_2 = u_3 = 0.25$.

Compare the 2-tuples a_1, a_2 and a_3, we have $a_2 > a_1 > a_3$. Thus,

$$\begin{array}{lll} ind(1) = 2, & ind(2) = 1, & ind(3) = 3; \\ a_{ind(1)} = (s_5, 0), & a_{ind(2)} = (s_3, 0), & a_{ind(3)} = (s_2, 0.83); \\ u_{ind(1)} = 0.25, & u_{ind(2)} = 0.5, & u_{ind(3)} = 0.25. \end{array}$$

Now we assume the scope of the aggregation is expressed by a weighting vector $w = (0.2, 0.3, 0.5)^T$ of TOWA. By Eq. (7), we get the function such that

$$f(x) = \begin{cases} 0.6x, & 0 \le x \le \frac{1}{3}; \\ 0.2 + 0.3(3x - 1), & \frac{1}{3} \le x \le \frac{2}{3}; \\ 0.5 + 0.5(3x - 2), & \frac{2}{3} \le x \le 1. \end{cases}$$

Since $R_0 = 0,\ R_1 = 0.25,\ R_2 = 0.75,\ R_3 = 1$, we have $f(R_0) = 0,\ f(R_1) = 0.15,\ f(R_2) = 0.625,\ f(R_3) = 1$. Then by Eq. (8), we get the modified weights $v_1 = 0.15,\ v_2 = 0.475,\ v_3 = 0.375$.

Using the PTOWA operator, we get the overall satisfaction $(\tilde{b}, \tilde{\alpha})$ of alternative x:
$(\tilde{b}, \tilde{\alpha}) = \text{PTOWA}(a_1, a_2, a_3) = \Delta\left(\sum_{j=1}^{3} v_j \times \Delta^{-1}\left(a_{ind(j)}\right)\right) = \Delta(3.236) = (s_3, 0.236)$.

4 Concluding

For linguistic aggregation problems where there exists a prioritization relationship between the attributes, we propose a prioritized 2-tuple ordered weighted averaging (PTOWA) operator. Based on the PTOWA operator and the TOWA operator, we give a method to aggregate the satisfactions of attributes for an alternative. A numerical example is given to illustrate the feasibility and effectiveness of the proposed method.

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